MTH31 Test 2 Review.

p. 190 True False

№2 true (Sum rule)

№10 $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ False

second derivative

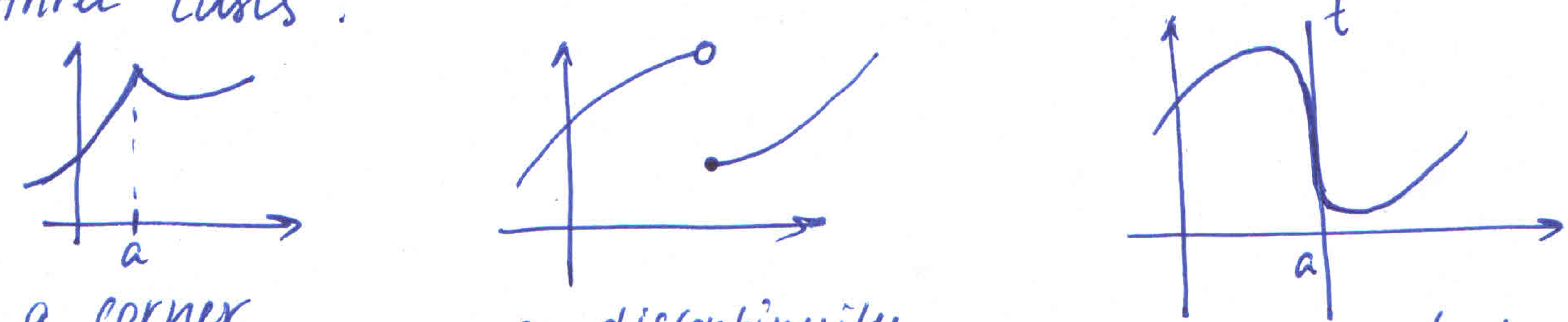
first derivative, squared.

p. 191 Exercises.

№2 f is not differentiable at $-4, -1, 2, 5$

recall that function is not differentiable in the below three cases:

$y = |x|$




a corner   a discontinuity   a vertical tangent

at $-4$ we have a discontinuity
at $-1$ we have a corner
at $2$ : discontinuity
at $5$ : vertical tangent.

№6 find a function f and a number **a** such that

$$\lim_{h \to 0} \frac{(2+h)^6 - 64}{h} = f'(a)$$

Solution: $\boxed{f(x) = x^6 \qquad a = 2}$

to check: $f(2) = 2^6 = 64$ ✓

comment: $f(a+h) = (2+h)^6$ and $f(a) = 64$

$\boxed{N10}$ $f(x) = \frac{4-x}{3+x}$ using definition.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{4-(x+h)}{3+(x+h)} - \frac{4-x}{3+x}}{h} =$$

$$= \lim_{h\to 0} \frac{(4-(x+h))(3+x) - (4-x)(3+x+h)}{h(3+x)(3+x+h)} =$$

$$= \lim_{h\to 0} \frac{4(3+x) - (x+h)(3+x) - (12+4x+4h-3x-x^2-hx)}{h(3+x)(3+x+h)} =$$

$$= \lim_{h\to 0} \frac{\cancel{12}+\cancel{4x}-\cancel{3x}-\cancel{x^2}-3h-\cancel{hx}-\cancel{12}-\cancel{4x}-4h+\cancel{3x}+\cancel{x^2}+\cancel{hx}}{h(3+x)(3+x+h)} =$$

$$= \lim_{h\to 0} \frac{-3h-4h}{h(3+x)(3+x+h)} = \lim_{h\to 0} \frac{-7\cancel{h}}{\cancel{h}(3+x)(3+x+h)} = \boxed{\frac{-7}{(3+x)^2}}$$

$\boxed{N11}$ $f(x) = x^3+5x+4$ using definition.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3+5(x+h)+4-(x^3+5x+4)}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3+\cancel{5x}+5h+\cancel{4}-\cancel{x^3}-\cancel{5x}-\cancel{4}}{h} =$$

$$= \lim_{h\to 0} (3x^2+3xh+h^2+5) = \boxed{3x^2+5}$$

$\boxed{N42}$ If $g(\theta) = \theta \sin\theta$, find $g''\left(\frac{\pi}{6}\right)$ comment: $\theta$ is read as "theta"

Solution:

$$g'(\theta) = \sin\theta + \theta\cos\theta$$

$$g''(\theta) = \cos\theta + \cos\theta + \theta(-\sin\theta) = \boxed{2\cos\theta - \theta\sin\theta.}$$

$\boxed{N44}$ Find $f^{(n)}(x)$ if $f(x) = \frac{1}{2-x}$.

Solution: $f(x) = (2-x)^{-1}$

$$f'(x) = (-1)(2-x)^{-2}\cdot(-1) = (2-x)^{-2} = \boxed{\frac{1}{(2-x)^2} = f'(x)}$$

$$f''(x) = (-2)(2-x)^{-3}\cdot(-1) = 2(2-x)^{-3} = \boxed{\frac{2}{(2-x)^3} = f''(x)}$$

$$f'''(x) = 2(-3)(2-x)^{-4}(-1) = 6(2-x)^{-4} = \boxed{\frac{6}{(2-x)^4} = f'''(x)}$$

$$f^{(4)}(x) = 6(-4)(2-x)^{-5}\cdot(-1) = 24(2-x)^{-5} = \boxed{\frac{24}{(2-x)^5} = f^{(4)}(x)}$$

conclusion: $\boxed{f^{(n)}(x) = \frac{n!}{(2-x)^{n+1}}}$

$\boxed{N48}$ $y = \frac{x^2-1}{x^2+1}$, $(0,-1)$ find an equation of tangent line.

Solution: $$y' = m = \frac{2x(x^2+1) - (x^2-1)\cdot 2x}{(x^2+1)^2} = \frac{\cancel{2x^3} + 2x - \cancel{2x^3} + 2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

$m_{(0,-1)} = 0$ - horizontal line.

$$\boxed{y_{\text{tangent}} = -1}$$

$\boxed{N50}$ $x^2+4xy+y^2=13$ at $(2,1)$ - find equations of tangent and normal lines.

Solution: we have implicitely defined function, so let's use implicit differentiation:

$$\frac{d}{dx}(x^2+4xy+y^2)=\frac{d}{dx}13.$$

$$2x+4y+4xy'+2yy'=0$$

$$y'(4x+2y)=-(2x+4y)$$

$$\boxed{y'=-\frac{x+2y}{2x+y}.}$$

Hence, the slope of the tangent line

$$m_{\substack{\text{tangent}\\ \text{line}}}=-\frac{2+2}{4+1}=-\frac{4}{5}$$

$$y-1=-\frac{4}{5}(x-2)$$

$$\boxed{y_{\text{tangent line}}=-\frac{4}{5}x+\frac{13}{5}}$$

the slope of the normal line is the negative reciprocal of the slope of the tangent line:

$$m_{\substack{\text{normal}\\ \text{line}}}=\frac{5}{4}$$

$$y-1=\frac{5}{4}(x-2)$$

$$-\frac{10}{4}+1=\frac{-6}{4}=-\frac{3}{2}$$

$$\boxed{y_{\substack{\text{normal}\\ \text{line}}}=\frac{5}{4}x-\frac{3}{2}}$$

$\boxed{N53}$ $y=\sin x+\cos x$, $0\le x\le 2\pi$

At what points on the curve is the tangent line horizontal?

Solution: there are two ways to proceede:

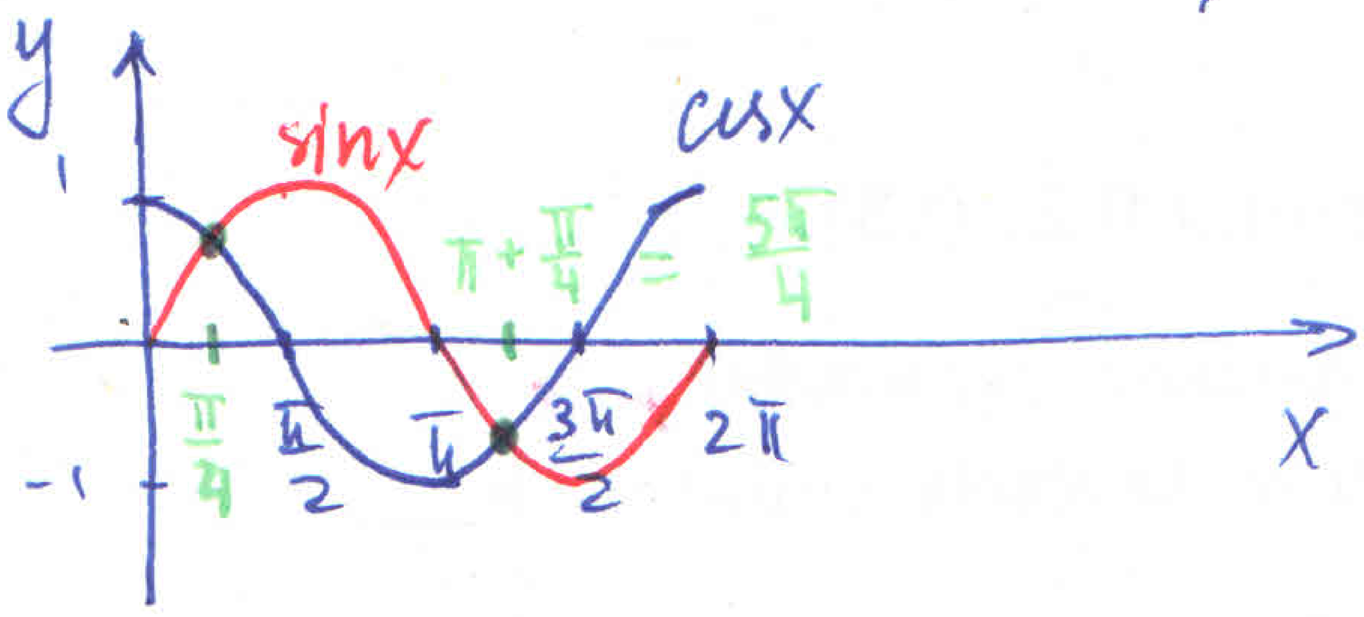
1) graph $y$

2) find the derivative $y'$ and solve $y'=0$

Let's try the second way:

$y' = \cos x - \sin x \qquad y' = 0$, i.e. $\cos x - \sin x = 0$

i.e. $\cos x = \sin x \quad , \quad 0 \le x \le 2\pi$




$\boxed{x = \frac{\pi}{4}, \ \pi + \frac{\pi}{4}}$

$\boxed{y = \frac{2\sqrt{2}}{2} = \sqrt{2} \ , -\sqrt{2}}$ $\Rightarrow$ $\boxed{\left(\frac{\pi}{4}, \sqrt{2}\right), \left(\frac{5\pi}{4}, -\sqrt{2}\right)}$

$\boxed{N54}$ Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

Solution: 1) $y' = m$ of tangent line

2) use implicit differentiation:

$\frac{d}{dx}(x^2 + 2y^2) = \frac{d}{dx} 1 \qquad 2x + 4yy' = 0$

$y' = -\frac{2x}{4y} \qquad y' = -\frac{x}{2y}$.

3) $-\frac{x}{2y} = 1$ , i.e. $-x = 2y$ , i.e. $\boxed{y = -\frac{1}{2}x}$

all the points on this straight line, that belong to the ellipse are the points at which the tangent line has slope 1.

4) $\begin{cases} x^2 + 2y^2 = 1 \\ y = -\frac{1}{2}x \end{cases}$

$x^2 + 2 \cdot \left(-\frac{1}{2}x\right)^2 = 1$

$x^2 + 2 \cdot \frac{1}{4}x^2 = 1$

$\frac{3}{2}x^2 = 1$

$x^2 = \frac{2}{3} \qquad x = \pm\sqrt{\frac{2}{3}}$ , hence $y = \mp \frac{1}{2} \cdot \sqrt{\frac{2}{3}} = \mp \frac{1}{\sqrt{6}}$

$\boxed{\left(\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}\right), \left(-\sqrt{\frac{2}{3}}, +\frac{1}{\sqrt{6}}\right)}$

$\boxed{N58}$ (a) $\cos 2x = \cos^2 x - \sin^2 x$

$$\frac{d}{dx}\cos 2x = \frac{d}{dx}(\cos^2 x - \sin^2 x)$$

$$-\sin 2x \cdot 2 = 2\cos x(-\sin x) - 2\sin x \cos x$$

$$2\sin 2x = 2\cos x \sin x + 2\sin x \cos x$$

$$2\sin 2x = 4\cos x \sin x$$

$$\boxed{\sin 2x = 2\cos x \sin x}$$

b) $\sin(x+\alpha) = \sin x \cos\alpha + \cos x \sin\alpha$ $\alpha$ - "alpha" - constant (doesn't depend on $x$)

$$\frac{d}{dx}(\sin(x+\alpha)) = \frac{d}{dx}(\sin x \cos\alpha + \cos x \sin\alpha)$$

$$\cos(x+\alpha) = \cos x\cos\alpha + \cancel{\sin x \cdot (0)} + (-\sin x)\sin\alpha + \cancel{\cos x \cdot (0)}$$

$$\boxed{\cos(x+\alpha) = \cos x \cos\alpha - \sin x \sin\alpha}$$

$\boxed{N60}$ $P(x) = f(x)g(x)$ $C(x) = f(g(x))$

$Q(x) = \dfrac{f(x)}{g(x)}$

a) $P'(2) = \underset{-1}{f'(2)}\,\underset{4}{g(2)} + \underset{1}{f(2)}\,\underset{1}{g'(2)} = -4+1 = \boxed{-3}$

b) $Q'(2) = \dfrac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{-4-1}{4^2} = \boxed{\dfrac{-5}{16}}$

c) $C'(2) = f'(\overbrace{g(2)}^{=4}) \cdot g'(2) = 1 \cdot 1 = \boxed{1}$

Comment: $C'(x) = \dfrac{d}{dx} f(g(x)) = \quad = f'(g(x)) \cdot g(x)$

N 76 $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$ ($)

a) The marginal cost function:

$$\boxed{C'(x) = 2 - 0.04x + 0.00021x^2 \quad (\$)}$$

b) $C'(100) = 2 - 4 + 2.1 = 0.1$ $

the cost of producing $101^{th}$ unit is 10 cents.

c) $C(100) = 920 + 200 - 200 + 70 = 990$ $

$C(101) = 920 + 202 - 204.02 + 72.12107 = 990.10107$ $

$C(101) - C(100) = 0.10107$ $

10.107 cents

N 87 $\lim\limits_{h \to 0} \dfrac{\sqrt[4]{16+h} - 2}{h}$

Solution: we'll match it with $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

$a = 16$ $f(x) = \sqrt[4]{x}$

for check: $f(16) = \sqrt[4]{16} = 2$

So we are differentiating $f(x) = \sqrt[4]{x}$ at 16.

$f'(x) = \frac{1}{4} x^{-\frac{3}{4}}$ $f'(16) = \frac{1}{4} \cdot 16^{-\frac{3}{4}} = \dfrac{1}{4\sqrt[4]{16^3}} = \dfrac{1}{4 \cdot 2^3} = \boxed{\dfrac{1}{32}}$

N 88 $\lim\limits_{\theta \to \pi/3} \dfrac{\cos\theta - 0.5}{\theta - \pi/3}$

Solution: we'll match it with $f'(a) = \lim\limits_{\theta \to a} \dfrac{f(\theta) - f(a)}{\theta - a}$

Hence, $f(x) = \cos\theta$, $a = \pi/3$, for check: $f(\frac{\pi}{3}) = \cos\frac{\pi}{3} = 0.5$

So, $f'(x) = -\sin\theta$ and $f'(\frac{\pi}{3}) = -\sin\frac{\pi}{3} = \boxed{-\dfrac{\sqrt{3}}{2}}$